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Instantons, compactification and S-duality in $\mathcal{N}=4$ SUSY Yang-Mills theory I

Nick Dorey

Department of Physics, University of Wales Swansea
Singleton Park, Swansea, SA2 8PP, UK
E-mail: n.dorey@swansea.ac.uk

Abstract: We study $\mathcal{N}=4$ supersymmetric Yang-Mills (SYM) theory with gauge group SU(2) compactified to three dimensions on a circle of circumference β. The eight fermion terms in the effective action on the Coulomb branch are determined exactly, for all β, assuming the existence of an interacting Spin(8) invariant fixed point at the origin. The resulting formulae are manifestly invariant under the $\mathrm{SL}(2,\mathbb{Z})$ duality of four-dimensional $\mathcal{N}=4$ SYM and lead to interesting quantitative predictions for instanton effects in gauge theory and in type-II string theory.

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Gauge theories with extended supersymmetry generically have a continuous degeneracy of vacua and can be analysed by constructing a wilsonian effective action for the corresponding massless fields. Typically the lowest non-trivial terms in the derivative expansion of the effective action are highly constrained by supersymmetry and can sometimes be determined exactly. In the following we will study $\mathcal{N}=4$ supersymmetric Yang-Mills theory with gauge group SU(2) compactified to three dimensions on a circle of circumference β. As in other theories with sixteen supercharges [1], non-trivial quantum effects first appear in terms with four spacetime derivatives. The supersymmetric completion of the effective action also involves terms with eight fermions and no spacetime derivatives. In this letter, building on previous work in the three-dimensional case [2, 3, 4], we will determine these terms exactly and extract predictions for interesting non-perturbative effects in gauge theory and in type-II string theory.

The approach, based on that of [4], is to formulate the condition for unbroken supersymmetry as a Laplace equation on the vacuum moduli space. The relevant solution of the Laplace equation is further constrained by its behaviour near the origin of the moduli space. The key assumption here is that the theory at this singular point is an interacting three-dimensional superconformal field theory with Spin(8) R-symmetry[1] [1, 5, 6]. The symmetries of this SCFT determine the behaviour of the eight fermion term near the origin. The exact eight fermion term can then be reconstructed using elementary facts about the uniqueness of harmonic functions. Its notable that the resulting term in the effective action is automatically invariant under the $\mathrm{SL}(2,\mathbb{Z})$ electric-magnetic duality of the four-dimensional theory. This reinforces the connection between S-duality in four dimensions and the Spin(8) R-symmetry in three dimensions discussed in [1].

At weak coupling, our result has two distinct instanton expansions, valid near the three- and four-dimensional limits, respectively. The model can be realized on the world volume of two D-branes in type-II string theory on $\mathbb{R}^9 \times S^1$. In this context, the field theory instanton effects exhibit many of the characteristic properties of type-II string theory instantons obtained by wrapping D-branes around non-trivial cycles in spacetime. In fact, the two expansions have an interpretation in terms of wrapped branes of the IIA and IIB theories, respectively and are related by T-duality. In a forthcoming paper [7], we investigate these effects via a direct semiclassical calculation and relate them to the L^2 index theory which counts BPS monopole/dyon boundstates in four dimensions [8].

[1] As discussed further below, our approach differs slightly from that of [4]. In particular, these authors were able to obtain the exact eight fermion term in three dimensions *without* assuming Spin(8) R-symmetry. Generalizing their derivation to the theory on $\mathbb{R}^3 \times S^1$ requires some additional technical assumptions. We will avoid this complication by retaining the assumption of Spin(8) R-symmetry at the conformal point.

We begin by considering $\mathcal{N}=4$ SUSY Yang-Mills theory in four-dimensional Minkowski space with gauge group $G=\mathrm{SU}(2)$. The theory contains six real scalar fields which transform in the adjoint representation of the gauge group and the **6** of the R-symmetry group $\mathrm{Spin}(6)_{\mathcal{R}} \equiv \mathrm{SU}(4)_{\mathcal{R}}$. Non-zero vacuum expectation values (VEVs) for these fields break the gauge group down to U(1). After taking into account the Higgs mechanism, the massless fields lie in a single multiplet of $\mathcal{N}=4$ SUSY which contains the massless U(1) gauge field A_μ, with four-dimensional Lorentz index $\mu=0,1,2,3$. We define abelian electric and magnetic fields of the low energy theory $E_i=F_{0i}=\partial_0 A_i-\partial_i A_0$ and $B_i=\varepsilon_{ijk}F^{jk}/2$ with $i=1,2,3$. The superpartners of the gauge field include neutral Weyl fermions λ^M_α and $\bar{\lambda}^M_{\dot{\alpha}}$, with $M=1,2,3,4$, in charge conjugate spinor representations of $\mathrm{Spin}(6)_{\mathcal{R}}$ as well as neutral scalars ϕ_a, with $a=1,\ldots,6$ in the vector representation of this group. We define a $\mathrm{Spin}(6)_{\mathcal{R}}$ invariant VEV $|\phi|$ with $|\phi|^2=\sum_{a=1}^{6}\phi_a^2$. After taking into account the Weyl group, which acts as $\phi_a \to -\phi_a$, the massless scalars parametrize a classical Coulomb branch which is $\mathbb{R}^6/\mathbb{Z}_2$.

The spectrum of the theory on the Coulomb branch includes BPS states which carry electric and magnetic charges with respect to the unbroken U(1) gauge symmetry. In terms of the low energy fields these are defined as,

$$q=\frac{1}{g^2}\int d^3x\vec{\nabla}\cdot\vec{E} \tag{1}$$

and

$$k=\frac{1}{4\pi}\int d^3x\vec{\nabla}\cdot\vec{B}\,, \tag{2}$$

respectively. The normalization of the electric charge is chosen so that a massive vector boson has charge +1. The Dirac quantization condition then implies that k is an integer.[2] Alternatively, in the full non-abelian theory, the magnetic charge, k is identified with a non-trivial element of the homotopy group $\pi_2(\mathrm{SU}(2)/\,\mathrm{U}(1))$ which yields the same integer quantization of k.

The four-dimensional theory has a single parameter, the complexified coupling $\tau=4\pi i/g^2+\theta/2\pi$ and is believed to be invariant under $\mathrm{SL}(2,\mathbb{Z})$ group of S-duality transformations [9] which act on τ as $\tau\to(a\tau+b)/(c\tau+d)$ for integers a,b,c,d with $ad-bc=1$. In the following $M(a,b,c,d)$ denotes the corresponding element of $\mathrm{SL}(2,\mathbb{Z})$,

$$M=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\mathrm{SL}(2,\mathbb{Z})\,. \tag{3}$$

The spectrum of theory includes BPS states which lie in short representations of the supersymmetry algebra. The states carry the electric and magnetic charges defined

[2]Strictly speaking, the Dirac quantization condition only requires k to be half integer. As usual the consistency of the theory in the presence of charges in the fundamental representation of the gauge group further restricts k to integer values

above, and their masses are determined in terms of these quantum numbers to be,

$$M(q,k) = |\phi||k\tau + q| = |\phi|\sqrt{k^2\left(\frac{4\pi}{g^2}\right)^2 + \left(q + k\frac{\theta}{2\pi}\right)^2}. \tag{4}$$

The two component charge vector $\vec{q}$ with $q_1 = q$, $q_2 = k$ transforms as $\vec{q} \to \tilde{M}\cdot\vec{q}$ under the $\mathrm{SL}(2,\mathbb{Z})$ transformation $M(a,b,c,d)$ defined above, where $\tilde{M} = M(a,-b,-c,d)$. If we assign the modular weight[3] $(1/2, 1/2)$ to the scalar fields ϕ_a then the BPS mass formula (4) is invariant under S-duality. However S-duality of the spectrum also requires that a single particle BPS state exists whenever q and k are coprime [8].

The four-dimensional Lorentzian theory with gauge group SU(2) can be realized on the world volume of two D3-branes in type-IIB string theory on flat $R^{9,1}$. Moving onto the Coulomb branch parametrized by the six scalar fields of the theory corresponds to seperating the branes in their six transverse directions. The distance between the D3-branes is proportional to the $\mathrm{Spin}(6)_{\mathcal{R}}$ invariant VEV $|\phi|$ (in units of α': we will henceforth set the string length scale $\sqrt{\alpha'}$ to unity). The IIB theory has a complex coupling τ which is identified with the complexified gauge coupling of the $\mathcal{N}=4$ theory and an S-duality group which corresponds to the $\mathrm{SL}(2,\mathbb{Z})$ invariance of the $\mathcal{N}=4$ theory on the D3-brane world volume. The IIB theory contains (q,k)-strings with tensions proportional to $|k\tau + q|$ for all coprime values of q and k. The BPS states discussed above are realized in string theory as configurations where a (q,k)-string is stretched between the two D3-branes. The first equality in (4) shows that the mass of each BPS state is just the tension of the corresponding string multiplied by the distance between the two branes.

We now switch our attention to the euclidean theory and also compactify the euclidean time direction, x_0, on a circle of circumference β. This breaks the four-dimensional euclidean group to its three-dimensional counterpart. The U(1) gauge field in four dimensions splits up into a scalar A_0 as well as a three-dimensional gauge field A_i with Lorentz index $i = 1,2,3$. The massless modes in three dimensions correspond to the Fourier modes of the fields with zero momentum in the compactified direction. Physically inequivalent configurations are defined up to gauge transformations $A_\mu \to A_\mu + \partial_\mu\chi(x)$. As all the fields in the underlying non-abelian gauge theory are invariant under the center of the gauge group, the necessary condition for such a gauge transformation to be single valued on $\mathbb{R}^3 \times S^1$ is $\chi(x_0+\beta) = \chi(x_0) + 2\pi n$. If $n \neq 0$ the resulting gauge transformation is topologically non trivial. The integer n is an element of $\pi_1(S^1) = Z$ which we will call the index of the gauge transformation. These so-called large gauge transformations will play an important role in the following.

[3] A modular form $f(\tau,\bar{\tau})$ of weight $(w,\bar{w})$ tranforms as $f \to (c\tau+d)^w(c\bar{\tau}+d)^{\bar{w}}f$ under the modular tranformation $T(a,b,c,d)$ discussed in the text.

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In addition to the four-dimensional scalars ϕ_a, gauge-inequivalent vacua are parametrized by VEV of the Wilson line $\omega = \int_{S^1} A \cdot dx$. The Wilson line is not invariant under the large gauge transformations discussed above but rather transforms as $\omega \to \omega + 2n\pi$. Hence this field should be thought of as a periodic variable with period 2π. As usual we can eliminate the abelian gauge field A_i in favour of a periodic scalar σ via a three-dimensional duality transformation. The field σ enters as a Lagrange multiplier for the Bianchi identity and leads to a surface term in the action, $S_\sigma = i\sigma k$ where k is the magnetic charge defined in (2) above. As all the remaining terms in the action depend on σ only via its spacetime derivatives, σ is effectively a periodic variable with period 2π. The manifold of gauge-inequivalent classical vacua parametrized by the VEVs of ϕ_a, ω and σ is $\mathcal{M}_{\rm cl} = (\mathbb{R}^6 \times \mathbb{T}^2)/\mathbb{Z}_2$.

Although the unbroken R-symmetry of the lagrangian is just the $\mathrm{Spin}(6)_{\mathcal{R}}$ of the four-dimensional theory, the resulting supersymmetry algebra with sixteen supercharges in three dimensions[4] has a larger $\mathrm{Spin}(8)_{\mathcal{R}}$ group of automorphisms. To make this manifest, we combine the eight scalar fields in a vector with components X_l, with $l = 1, \ldots, 8$ as:

$$\vec{X} = \left(\sqrt{\delta}\phi_1, \ldots, \sqrt{\delta}\phi_6, \sqrt{\gamma}\omega, \sqrt{\epsilon}\left(\sigma + \frac{\theta\omega}{2\pi}\right)\right) \tag{5}$$

with $\delta = \beta/g^2$, $\gamma = 1/g^2\beta$ and $\epsilon = g^2/(16\pi^2\beta)$. The enlarged R-symmetry is such that $\vec{X}$ transforms in the eight-dimensional vector representation $\mathbf{8}_V$ of $\mathrm{Spin}(8)_{\mathcal{R}}$. In the following Roman indices $l, m, n, o = 1, \ldots, 8$ label the components of this repesentation. The low energy theory also includes eight Majorana fermions ψ^Ω_α, with $\Omega = 1, \ldots, 8$ and $\alpha = 1, 2$, which comprise one of the two eight-dimensional spinor representations of $\mathrm{Spin}(8)_{\mathcal{R}}$ denoted $\mathbf{8}_S$. The SUSY algebra then takes the form,

$$\begin{aligned} \delta X^l &= -i\epsilon^\alpha_{\dot\Omega}\Gamma^l_{\dot\Omega\Sigma}\psi_{\alpha\,\Sigma}\,, \\ \delta\psi_{\alpha\Sigma} &= \epsilon^\beta_{\dot\Omega}\Gamma^l_{\dot\Omega\,\Sigma}\tau^i_{\beta\alpha}\partial_i X^l\,. \end{aligned}$$

Here $\Gamma^l_{\dot\Omega\Sigma}$ represent the Spin(8) Clifford algebra and the dotted and undotted indices $\dot\Omega$ and Σ are indices of $\mathbf{8}_C$ and $\mathbf{8}_S$ representations of $\mathrm{Spin}(8)_{\mathcal{R}}$, respectively. The τ_i are the γ-matrices for the spacetime Clifford algebra in three dimensions.

The bosonic part of the classical effective action is [10] $S^B_{\rm eff} + S_\sigma$, with,

$$S^B_{\rm eff} = \int d^3x\,\frac{1}{2}\,\delta_{lm}\partial_\mu X^l\partial^\mu X^m\,. \tag{6}$$

While the fermionic part is,

$$S^F_{\rm eff} = \int d^3x\,\frac{1}{2}\,\delta_{\Omega\Sigma}\psi^\Omega\tau^i\partial_i\psi^\Sigma\,. \tag{7}$$

[4]This is sometimes refered to as $N = 8$ SUSY in three dimensions. Here N counts the number of two-component Majorana supercharges. We prefer to adopt the four-dimensional convention and count the number $\mathcal{N}$ of complex two-component supercharges which is equal to four in this case.

This is the action of a three-dimensional supersymmetric non-linear σ-model whose target manifold is $\mathcal{M}_{\rm cl} = (\mathbb{R}^6 \times \mathbb{T}^2)/\mathbb{Z}_2$ with the standard flat metric. The θ-dependent shift in the definition (5) of X_8 reflects the fact that the four-dimensional θ-term is proportional to $\vec{E}\cdot\vec{B}$ and therefore yields a coupling between the spacetime derivatives ω and σ. Note that the Spin$(8)_{\mathcal{R}}$ symmetry is broken to Spin$(6)_{\mathcal{R}}$ solely by the periodic boundary conditions on the bosonic fields X_7 and X_8;

$$\begin{aligned} X_7 &\sim X_7 - 2n_2\Omega_2\,, \\ X_8 &\sim X_8 - 2n_1\Omega_1 - 2\kappa n_2\Omega_2 \end{aligned} \tag{8}$$

with $2\Omega_1 = g/2\sqrt{\beta}$, $2\Omega_2 = 2\pi/g\sqrt{\beta}$ and $\kappa = \theta g^2/8\pi^2$. The real two-dimensional torus parametrized by X_7 and X_8 corresponds to a complex torus E with complex structure parameter τ and holomorphic coordinate $Z = -i(\tau\omega+\sigma)$. Note that S-duality simply corresponds to invariance under modular transformations of the complex torus E [1]. If we define a two component vector $\vec{\sigma}$ with components $\sigma_1 = \sigma$ and $\sigma_2 = \omega$, then the transformation law $\vec{\sigma} \to \vec{\sigma}' = \tilde{M}\cdot\vec{\sigma}$ under $M(a,b,c,d) \in \mathrm{SL}(2,\mathbb{Z})$, with $\tilde{M} = M(a,-b-c,d)$ as above, ensures that Z transforms with modular weight $(-1,0)$. In the special case $\theta = 0$, where the torus is rectangular then the periodic boundary conditions (8) break Spin$(8)_{\mathcal{R}}$ to Spin$(6)_{\mathcal{R}}\times\mathbb{Z}_2$ where the $\mathbb{Z}_2$ factor is precisely the $\mathbb{Z}_2$ subgroup of $\mathrm{SL}(2,\mathbb{Z})$ generated by an electric-magnetic duality transformation [1].

The limit where the theory reduces to a gauge theory in three spacetime dimensions is obtained by taking $g^2 \to 0$ and $\beta \to 0$ with the three-dimensional gauge coupling $e^2 = 2\pi g^2/\beta$ held fixed. To get to a generic Coulomb phase vacuum of the $D = 3$ theory we must also take the Wilson line to zero, holding the three-dimensional scalar field $\phi_7 = \omega/\beta$ fixed. In this limit the period Ω_2 diverges while Ω_1 stays fixed. As one period of the torus decompactifies the Coulomb branch becomes $\mathbb{R}^7 \times S^1$ and the manifest R-symmetry group enlarges from Spin$(6)_{\mathcal{R}}$ to Spin$(7)_{\mathcal{R}}$. This is the theory analysed in [2, 3, 4]. Taking the alternative "strong coupling" limit $\beta \to 0$ with g^2 held fixed, both periods of the torus decompactify and the Coulomb branch becomes $\mathbb{R}^8$ with a manifest Spin$(8)_{\mathcal{R}}$ symmetry at the classical level.

In a theory with sixteen supercharges, supersymmetry does not allow any corrections to the classical metric on the Coulomb branch. The action given in (6) and (7) above, is therefore the exact effective action including terms with at most two spacetime derivatives or four fermions. Note that the Coulomb branch has $\mathbb{Z}_2$ orbifold singularities at the four fixed points of the Weyl group: $(X_7, X_8) = (0,0)$, $(\Omega_2, 0)$, $(0, \Omega_1 + \kappa\Omega_2)$ and $(\Omega_2, \Omega_1 + \kappa\Omega_2)$ with $X_l = 0$ for $l \leq 6$ in each case. The origin is distinguished from the other three fixed points because only at this point is the non-abelian gauge symmetry restored. As new light degrees of freedom appear at this point, we expect that the low-energy effective theory described above breaks down. In fact it is believed that the quantum theory at the origin is an interacting three-dimensional superconformal field theory with an unbroken Spin$(8)_{\mathcal{R}}$ [1, 5, 6].

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In contrast no new light degrees of freedom at the other fixed points and the low energy description remains valid at these points. Hence, apart from the SCFT at the origin, the theory in each vacuum on the Coulomb branch is free in the IR.

Many features of the compactified theory can be understood via its realization on the world volume of type-II branes. As above we start with two D3-branes of the type-IIB theory. However as we wish to discuss the euclidean version of $\mathcal{N} = 4$ SUSY Yang-Mills, we now consider the IIB theory on euclidean $\mathbb{R}^{10}$. After compactifying one of the directions along the D3 world volume we can perform a T-duality to the type-IIA theory compactified to nine dimensions on the dual circle. The D3-branes now become D2-branes located at a point on the S^1 factor. Moving the branes apart along the S^1 corresponds to introducing a non-zero Wilson line ω. The periodic nature of this variable is manifest from this point of view. As the dual photon is intrinsically quantum mechanical, the second compact direction does not appear directly in weakly coupled string theory. Instead we can invoke the duality between the type-IIA theory on $\mathbb{R}^9 \times S^1$ and M-theory on $\mathbb{R}^9 \times \mathbb{T}^2$. The D2-branes lift to M2-branes in eleven dimensions. The radius of the eleventh (or M-) direction is smaller by a power of g^2 than that of the other compact direction. The separation of the branes in this direction corresponds to the dual photon [11]. The $\mathrm{SL}(2,\mathbb{Z})$ invariance of the T-dual IIB theory and therefore of $\mathcal{N} = 4$ SYM follows naturally from invariance under modular transformation of the torus $\mathbb{T}^2$. This "geometrization" of IIB S-duality was first discovered in [12]. The fact that an analogous geometrical understanding of the Olive-Montonen duality of the $\mathcal{N} = 4$ theory emerges after toroidal compactification was anticipated in [13, 14].

The M-theory perspective also provides a natural explanation for the existence of a $\mathrm{Spin}(8)_{\mathcal{R}}$ invariant superconformal fixed point in the strong coupling limit [15]. This limit is precisely the decompactification limit for the M-dimension and the $\mathrm{Spin}(8)_{\mathcal{R}}$ invariance is just the euclidean group of spacetime rotations acting on the eight dimensions transverse to the M2-branes. At least for gauge group $\mathrm{SU}(N)$ with N large, which corresponds to a large number of M2-branes, the resulting three-dimensional SCFT is believed to be dual to M-theory compactified on $AdS_4 \times S^7$ via the AdS/CFT correspondence [16]. Both the three-dimensional conformal group and $\mathrm{Spin}(8)_{\mathcal{R}}$ are manifest as isometries of this spacetime. Finally, the appearance of a $\mathrm{Spin}(8)_{\mathcal{R}}$ invariant fixed point also plays an important role in the Matrix model of M-theory [17]. In this context, it is necessary for an understanding of the limit of M-theory on $\mathbb{T}^2$ which yields IIB string theory on $\mathbb{R}^{10}$ [5].

As mentioned above, the two derivative terms in the effective action are classically exact. Non-trivial quantum corrections to the wilsonian effective action first appear at the next order in the derivative expansion which includes terms with four spacetime derivatives. The supersymmetric completion of these bosonic terms include terms with eight fermions and no spacetime derivatives and our main aim below will be to determine these terms exactly. These terms are generated in perturbation theory and also recieve instanton corrections which we will now discuss. The four-dimensional theory contains classical BPS monopoles of unit magnetic charge which appear as static solitons of mass $M(0,1) = (4\pi/g^2)|\phi|$. For each value of the magnetic charge k, the theory has a $4k$-parameter family of exact multi-monopole solutions of mass $M(0,k) = kM(0,1)$. After compactification these configuration become instantons of finite action euclidean action $\beta k M(0,1)$. Taking into account the magnetic surface term S_σ discussed above, these instantons yield corrections proportional to $\exp(-\beta k M(0,1) + ik\sigma)$ to the low-energy effective action. We will analyse these corrections, and other related effects in detail in the following and in [7].

Static BPS monopoles in four dimensions are invariant under half the supersymmetry algebra. It is often convenient to work in a formalism where this invariance is manifest. This is accomplished by first performing an $\mathrm{SU}(4)_{\mathcal{R}}$ rotation so that only one component of the Higgs field, say ϕ_1, is non zero. After choosing the gauge $A_0 = 0$, we then define an auxiliary euclidean four-dimensional gauge theory with an SU(2) gauge field $v_{\tilde{\mu}}$ with $v_{\tilde{\mu}} = A_i$ for $\tilde{\mu} = i = 1,2,3$ and $v_4 = \phi_1$. The easiest way to motivate this is to recall that $\mathcal{N} = 4$ SUSY Yang-Mills theory can be derived by dimensional reduction of the minimal supersymmetric gauge theory in ten dimensions. If our original four-dimensional theory is embedded in the x_0, x_i dimensions with $i = 1,2,3$ then the auxiliary euclidean theory is simply the corresponding copy $\mathcal{N} = 4$ SYM embedded in the x_i, x_4 directions. Clearly, classical solutions which depend only on x_i can be thought of as static configurations in either theory. In the original theory, ten-dimensional Lorentz invariance is broken to the product of the four-dimensional $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ Lorentz group and $\mathrm{Spin}(6)_{\mathcal{R}}$. The symmetries of the auxiliary theory correspond to a different embedding of these groups into Spin(10). We denote the corresponding subgroups, $\mathrm{SU}(2)_{\tilde{L}} \times \mathrm{SU}(2)_{\tilde{R}} \times \mathrm{Spin}(6)_{\tilde{\mathcal{R}}}$. The auxiliary theory contains four Weyl fermions of each chirality transforming as $(\mathbf{2},\mathbf{0},\mathbf{4})$ and $(\mathbf{0},\mathbf{2},\bar{\mathbf{4}})$, respectively under these symmetries. We denote these ρ^A_δ and $\bar{\rho}^A_{\dot{\delta}}$ with corresponding indices $\delta = 1,2$ and $\dot{\delta} = \dot{1},\dot{2}$ and $A = 1,2,3,4$.

The advantage of the above construction is that the original Bogomol'nyi equation for A_i and ϕ can be rewitten as the self-dual Yang-Mills equation for the auxiliary gauge field. Thus the magnetic instanton is manifestly invariant under supercharges of one four-dimensional chirality. We emphasize that this refers to the chirality of Weyl spinors in the auxiliary theory described above and not in the original four-dimensional theory. The action of the remaining supercharges generates eight fermion zero modes of the instanton. The Callias index theorem [18] indicates that the instanton solution has many additional zero modes. However, as discussed in [3], all of these are lifted by couplings to the scalar fields of the theory. The instanton therefore has eight exact fermion zero modes of the same chirality it contributes to an eight fermion term in the action built only out of the right-handed Weyl fermions $\bar{\rho}^A_{\dot{\delta}}$. As these fields have only eight independent components the form of the vertex is uniquely determined to be product of these eight and can be written as,

$$\mathcal{L}_{8f} = \mathcal{V}(\sigma, \omega, |\phi|) \prod_{A=1}^{4} \bar{\rho}^A_{\dot{\delta}} \bar{\rho}^{A\,\dot{\delta}} \,. \tag{9}$$

The vertex (9) contributes to the large distance behaviour of the eight fermion correlation function,

$$\mathcal{G}^{(8)}(x_1, \ldots, x_8) = \left\langle \prod_{A=1}^{4} \rho^A_1(x_{2A}) \rho^A_2(x_{2A-1}) \right\rangle , \tag{10}$$

Strictly speaking, as we started our discussion of the auxiliary theory by setting $A_0 = 0$, the above result only applies when the Wilson line ω is set to zero. A more complete discussion of the instanton contribution will be given in [7]. Its also important to note that the chirality selection rule apparant in (9) is only valid at leading semiclassical order in a static background. In particular perturbative corrections in the instanton background can (and do) lead to other eight fermion structures which mix auxiliary fermions ρ^A_{δ} and $\bar{\rho}^A_{\dot{\delta}}$ of both chiralities. This simply reflects the fact that the auxiliary four-dimensional Lorentz group $\mathrm{SU}(2)_{\tilde{L}} \times \mathrm{SU}(2)_{\tilde{R}}$ is not an exact symmetry of the theory.

We will now give an exact analysis the eight fermion terms in the effective action. To begin with we write down the most general eight fermion term consistent with the symmetries of the theory. The various eight fermion stuctures which can arise can be organised according to their transformation properties under $\mathrm{Spin}(6)_{\mathcal{R}}$. As above we can define complex linear combinations of the three-dimensional Majorana fermions which correspond to Weyl fermions in four dimensions. These combinations transform in the $\mathbf{4}$ and $\bar{\mathbf{4}}$ of $\mathrm{Spin}(6)_{\mathcal{R}} \simeq \mathrm{SU}(4)_{\mathcal{R}}$. In general, we must expand all possible products of eight fermions, each of which can be either in the $\mathbf{4}$ or $\bar{\mathbf{4}}$ of $\mathrm{Spin}(6)_{\mathcal{R}}$, as a sum of irreducible representations of this group. As this is an unbroken symmetry of the theory we must then form $\mathrm{Spin}(6)_{\mathcal{R}}$ invariants by contracting these tensors with products of the scalar fields ϕ_a, each of which carry a vector index of $\mathrm{Spin}(6)_{\mathcal{R}}$. Hence for each integer L, we need only consider eight fermion structures which transform as rank-L symmetric traceless tensors which we denote $T^{(L)}_{a_1 \ldots a_L}(\psi)$. We write the most general possible coupling as,

$$\mathcal{L}_{8f} = \sum_{L} \mathcal{F}^{(L)}(\vec{X}) X_{a_1} \cdots X_{a_L} \, T^{(L)}_{a_1 \ldots a_L}(\psi) \,, \tag{11}$$

where X_a denotes the components of the $\mathrm{Spin}(8)_{\mathcal{R}}$ vector X_l introduced above with $l = a = 1, \ldots, 6$. In the following we will see that the maximum value of L which contributes to the sum is four. The above expansion is complicated by the fact that several linearly independent tensor structures may appear for each value of L, each of which can appear multiplied by a distinct scalar function of $\vec{X}$. Fortunately, our analysis will not be affected by this subtlety and we have supressed it in (11).

The supersymmetric variation of the bosonic fields appearing in the coefficients of the eight fermion terms leads to terms with nine fermions. Because all other terms appearing in the supersymmetric completion of the four derivative terms have fewer than eight fermions to start with, there is no other source for nine fermion terms in $\delta\mathcal{L}$. It follows that the nine fermion terms $\delta\mathcal{L}_{8f}$ must vanish and this constrains the allowed functions $\mathcal{F}^{(L)}(\vec{X})$. This constraint was studied in detail in [4] and we will use their result which adapts easily to the present case. The result is simply that the eight fermion lagrangian $\mathcal{L}_{8f}$ must be a harmonic function on the the Coulomb branch parameterized by $\vec{X}$. In other words we must have $\Delta(\mathcal{L}_{8f})$=0 where $\Delta = \sum_{l=1}^{8} \partial^2/\partial X_l^2$ is the laplacian on $\mathcal{M}_{\rm cl} = (\mathbb{R}^6 \times \mathbb{T}^2)/\mathbb{Z}_2$.

The key idea in the following will be the fact that harmonic functions, like holomorphic functions, are essentially determined by their behaviour at singularities (and at infinity). Hence if we can understand the behaviour of the effective action near its singular points we may reconstruct it everywhere on the moduli space. In this sense the argument to follow is a direct generalization of the arguments leading to the holomorphic superpotential for the $\mathcal{N}=1^*$ theory derived in [19]. In fact, when the Coulomb branch is two dimensional as it was in that case, a harmonic function is simply the real part of a holomorphic function.

As usual, singularities may appear at points where additional degrees of freedom become light. As mentioned above, it is believed that the only such point is the origin $\vec{X}=0$ where the non-abelian gauge symmetry is restored and the theory becomes superconformally invariant [1]. In the following we will assume this is true and use it to constrain the exact eight fermion terms in the action. The existence of an interacting SCFT at the origin implies that the effective action near the origin of the Coulomb branch should be invariant under scale transformations and $\mathrm{Spin}(8)_{\mathcal{R}}$ rotations. The latter assumption greatly simplifies the analysis of the eight fermion term. As noted above, our approach differs from the analysis of the three-dimensional theory presented in [4]. In particular they did not assume the presence of a $\mathrm{Spin}(8)_{\mathcal{R}}$ invariant fixed point at the origin but rather proved this directly from weaker starting assumptions. However, the situation in the compactified four-dimensional theory is more complicated because the unbroken R-symmetry on the Coulomb branch is only $\mathrm{Spin}(6)_{\mathcal{R}}$ compared to the $\mathrm{Spin}(7)_{\mathcal{R}}$ which appears in the three-dimensional limit. It seems possible that our approach can be adapted to yield the same results from weaker assumptions but we will not pursue this here.

In the limit $\vec{X}\to 0$, we may expand the product of eight fermions in symmetric, traceless rank-M tensor representations of $\mathrm{Spin}(8)_{\mathcal{R}}$, which we denote $U^{(M)}_{l_1\ldots l_M}(\psi)$ and write the most general $\mathrm{Spin}(8)_{\mathcal{R}}$ invariant vertex,

$$\mathcal{L}_{8f} \longrightarrow \sum_{M} \mathcal{G}^{(M)}(|\vec{X}|) X_{l_1}\ldots X_{l_M}\, U^{(M)}_{l_1\ldots l_M}(\psi)\,. \tag{12}$$

As above there can, in principle, be more than one Lorentz scalar, eight fermion tensor structure for each M. The Laplace equation $\Delta\mathcal{L}_{8f} = 0$, decomposes into decoupled linear equations for each function $\mathcal{G}^M(|\vec{X}|)$ each of which has a unique $\mathrm{Spin}(8)_{\mathcal{R}}$ invariant solution proportional to $|\vec{X}|^{-6-2M}$. Finally we must consider the restrictions placed by scale invariance. The fields $\vec{X}$ and ψ, lie in a chiral primary multiplet of the superconformal algebra and their classical scaling dimensions are not corrected by quantum effects. Thus we have $[X] = 1/2$ and $[\psi] = 1$ and the action is only scale invariant if $[\mathcal{G}^{(M)}] = -(M+10)/2$. On the other hand the solutions of the Laplace equation found above have dimension $-3-M$. Hence scale invariance uniquely selects the $M = 4$ term. In fact this is exactly the same eight fermion term which appears in the three-dimensional analysis of [4].

One can also check directly that the instanton induced vertex (9), corresponds to part of the tensor $U^{(4)}_{l_1\ldots l_4}$ and has no overlap with the $M < 4$ terms. To see this we note that the $\mathrm{Spin}(8)_{\mathcal{R}}$ symmetry of the conformal point includes as a subgroup the $\mathrm{Spin}(6)_{\tilde{\mathcal{R}}}$ symmetry of the auxiliary four-dimensional theory discussed above. Clearly $\mathrm{Spin}(8)_{\mathcal{R}}$ also includes an additional abelian symmetry, which we will denote $U(1)_N$, that commutes with $\mathrm{Spin}(6)_{\tilde{\mathcal{R}}}$. The Weyl fermions of the auxiliary theory, ρ^a_δ and $\bar{\rho}_{\dot{\delta}}$, are formed from complex linear combinations of the three-dimensional Majorana fermions ψ^a_α. These fields transform $(+1/2, \mathbf{4})$ and $(-1/2, \bar{\mathbf{4}})$ under $U(1)_N \times \mathrm{Spin}(6)_{\tilde{\mathcal{R}}}$, respectively. Hence, the product of eight right-handed fermions appearing in the instanton induced vertex (9) has the minimum possible $U(1)_N$ charge $Q_N = -4$. Its easy to check that the rank-M symmetric traceless tensor representation of $\mathrm{Spin}(8)_{\mathcal{R}}$ contains states of $U(1)_N$ charge $-M \le Q_N \le +M$. Thus the maximum value of M which can appear in (12) is $M = 4$ and moreover the instanton vertex (9) must correspond to part of this structure.

In summary the exact behaviour of eight fermion term in the $\vec{X} \to 0$ limit is,

$$\mathcal{L}_{8f} \longrightarrow \frac{X_{l_1}\cdots X_{l_4}}{|\vec{X}|^{14}}\, U^{(4)}_{l_1\ldots l_4}(\psi)\,. \tag{13}$$

This expression is invariant under scale and $\mathrm{Spin}(8)_{\mathcal{R}}$ symmetries and also satisfies the Laplace equation near the origin of the Coulomb branch. Away from the origin the $\mathrm{Spin}(8)_{\mathcal{R}}$ symmetry will be broken to $\mathrm{Spin}(6)_{\mathcal{R}}$ by the periodic boundary conditions on X_7 and X_8 and scale invariance will be broken by explicit dependence on the compactification radius. However supersymmetry still requires that $\mathcal{L}_{8f}$ satisfies the Laplace equation on $\mathcal{M}_{\mathrm{cl}} = (\mathbb{R}^6 \times \mathbb{T}^2)/\mathbb{Z}_2$ and this will suffice to determine it exactly. A harmonic function on $(\mathbb{R}^6 \times \mathbb{T}^2)/\mathbb{Z}_2$ defines a harmonic function on $\mathbb{R}^8/\mathbb{Z}_2$ which is invariant under the translations $\vec{X} \to \tilde{\vec{X}}$ with $\tilde{X}_l = X_l$ for $l \le 6$ and

$$\begin{aligned} \tilde{X}_7 &= X_7 - 2n_2\Omega_2\,, \\ \tilde{X}_8 &= X_8 - 2n_1\Omega_1 - 2\kappa n_2\Omega_2\,. \end{aligned} \tag{14}$$

The required harmonic function must have the behaviour (13) near the point $\vec{X} = 0$ and near each of its images under the above translations. As we are solving a linear equation, the required solution may be generated simply by summing 13) over n_1 and n_2 to get,

$$\mathcal{L}_{8f} = \sum_{n_1,n_2=-\infty}^{+\infty} \frac{\tilde{X}_{l_1}\cdots\tilde{X}_{l_4}}{|\vec{\tilde{X}}|^{14}} U^{(4)}_{l_1\ldots l_4}(\psi)\,. \tag{15}$$

Of course an important caveat is that this double sum is convergent. In the present case, the large power of $|\vec{\tilde{X}}|$ in the denominator ensures that this is the case. As we have specified the leading behaviour near each singularity, standard theorems [20] show that, provided this solution exists, it is unique. In the three-dimensional limit in which $\Omega_2 \to \infty$ with Ω_1 held fixed, we immediately reproduce the results of [4].

Comparing the expression (15), with our starting point (11) we may read off the exact expressions for the coefficient functions $\mathcal{F}^{(L)}(\vec{X})$ of each Spin$(6)_{\mathcal{R}}$ representation. We will focus on the highest value of L appearing in this sum, which is $L = 4$. We find,

$$\mathcal{F}^{(4)}(\vec{X}) = \sum_{n_1,n_2=-\infty}^{+\infty} \frac{1}{\left[(X_8 - 2n_1\Omega_1 - 2\kappa n_2\Omega_2)^2 + (X_7 - 2n_2\Omega_2)^2 + A^2\right]^7} \tag{16}$$

with $A^2 = \sum_{l=1}^{6} X_l^2 = \beta|\phi|^2/g^2$. Equation (16) has a nice interpretation in M-theory where the theory in question is realized on two M2-branes located a distance proportional to $(A^2 + X_7^2 + X_8^2)$ apart on their eight common transverse directions. Equation (16) can be a thought of as a sum over pairwise interactions between the two branes and all of their periodic images under the translations (14). This generalises the three-dimensional results of [2]. It can also be written in a form which is manifestly invariant under the SL$(2,\mathbb{Z})$ duality of the four-dimensional theory. In particular we have,

$$\mathcal{F}^{(4)}(\vec{X}) = \sum_{n_1,n_2=-\infty}^{+\infty} \frac{1}{\left[\frac{g^2}{16\pi^2\beta}|Z - n_2\tau - n_1|^2 + \frac{\beta}{g^2}|\phi|^2\right]^7}\,. \tag{17}$$

As above Z, $|\phi|$ and g^2 have modular weights $(-1,0)$, $(+1/2,+1/2)$ and $(+1,+1)$, respectively. The summand is modular if the two component vector $\vec{n} = (n_1,n_2)$ transforms as $\vec{n} \to \tilde{M}\cdot\vec{n}$ under $M(a,b,c,d) \in$ SL$(2,\mathbb{Z})$ with $\tilde{M} = M(a,-b,-c,d)$ as above. Hence modular transformations effectively permute different terms in the sum over n_1 and n_2. Modular invariance of $\mathcal{F}^{(4)}$ then follows after summing over n_1 and n_2.

In the remainder of the paper we will examine the physical content of the above result. First we extract the the perturbative part of the result which comes from the sector of zero magnetic charge,

$$\mathcal{F}^{(4)}_{\rm pert} = \left(\frac{g^2}{\beta}\right)^6 \sum_{n=-\infty}^{+\infty} \frac{1}{\left[\frac{(\omega-2\pi n)^2}{\beta^2} + |\phi|^2\right]^{13/2}}\,. \tag{18}$$

This term is entirely generated at one loop.[5] In the three-dimensional limit, $\beta \to 0$ and $g^2 \to 0$ with $e^2 = 2\pi g^2/\beta$ held fixed, only the $n = 0$ term in the sum contributes and we recover the behaviour $1/|\phi|^{13}$ of the one-loop correction computed in [4]. After Poisson resummation one may also take a four-dimensional limit which yields a one-loop term proportional to $1/|\phi|^{12}$, this is part of the supersymmetric completion of the one-loop $F^4_{\mu\nu}$ term in the effective action of $\mathcal{N} = 4$ SUSY Yang-Mills theory derived in [21].

It is straightforward to Fourier expand the result (16) in integer powers of $\exp(i\sigma)$ corresponding to sectors of different magnetic charge. Explicitly we obtain,

$$\mathcal{F}^{(4)}(\vec{X}) = \sum_{k=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \mathcal{F}_{k,n}(|\phi|, \omega) \exp\left(ik\left(\sigma + n\theta + \frac{\theta\omega}{2\pi}\right)\right). \tag{19}$$

In the weak coupling limit, $g^2 \ll 1$, the Fourier coefficient becomes,

$$\mathcal{F}_{k,n} \sim \beta^7 g^{-14} \frac{k^{13}}{S^7_{k,n}} \exp(S_{k,n}), \tag{20}$$

where,

$$S_{k,n} = \frac{4\pi k}{g^2} \sqrt{\beta^2|\phi|^2 + |\omega - 2\pi n|^2}. \tag{21}$$

Hence we have a sum of instanton contributions labelled by two integers k and n. To start with we note that, in the special case $\omega = 0$, the $n = 0$ terms yield the expected series of magnetic instanton corrections with exponential supression $\exp(-\beta k M(0, 1) + ik\sigma)$ where k is identified with the magnetic charge. Terms with $n \neq 0$ reflect the phenomenon described in [19, 22]: the magnetic instanton is not invariant under large gauge transformations. In fact the configurations with action $S_{k,n}$ given in (21) above, are each obtained by acting on the BPS monopole solution with a large gauge transformation of index n. Note that dependence on ω and n only arises in the combination $\omega - 2\pi n$, reflecting the fact that the Wilson line is shifted by $2\pi n$ under such gauge transformations. These instanton effects will be studied directly using semiclassical methods in [7]. The exact formula (16) also predicts a series of perturbative corrections to (20) suppressed by powers of g^2. Interestingly, this pertubative series truncates at a finite order in g^2.

The instantons described above have a simple interpretation in terms of wrapped branes in the IIA theory. As before our low energy theory lives on two D2-branes on $\mathbb{R}^9 \times S^1$ where the compact dimension is transverse to the brane world volume. The branes are seperated on S^1 by an amount ω/β and in one of the non-compact transverse directions by distance $|\phi|$. The IIA theory contains D0-branes with mass $4\pi/g^2$ (in units with $\sqrt{\alpha'} = 1$). The duality between the IIA string theory and M-theory, requires that any number of D0-branes form a bound state at threshold. These

[5]The presence of g^{12} as a prefactor reflects our choice of normalization for the fields

bound states are identified with the Kaluza-Klein modes of the eleven-dimensional metric and the number of constituent D0-brane corresponds to momentum in the M-direction. As explained above, in the present context, the VEV of the dual photon is naturally interpreted as the spatial coordinate in the eleventh direction. Noting that magnetic monopoles of charge k contribute to the path integral with a phase $\exp(ik\sigma)$ we immediately see that magnetic charge should be identified with D0-brane number. Each of the instantons of magnetic charge $\pm k$ described in preceeding paragraphs corresponds to a configuration where the euclidean worldline of a boundstate of k D0-branes is stretched between the two D2-branes. The possible paths between the two D2s split up into sectors labelled by an integer n which counts the number of times the D0 boundstate worldline winds round the compact dimension. The shortest path in each sector has length $\sqrt{\beta^2|\phi|^2 + |\omega - 2\pi n|^2}$ and the corresponding configuration therefore has action $S_{k,n} = (4\pi k/g^2)\sqrt{\beta^2|\phi|^2 + |\omega - 2\pi n|^2}$ in agreement with (21). In view of this correspondence we will refer to the instanton expansion (19) and (20) as the IIA expansion. A three-dimensional limit in field theory corresponds to de-compactifying the S^1 direction in the IIA picture while keeping the distance between the D2-branes fixed. Clearly only the configurations with winding number $n = 0$ contribute in this limit.

Its notable that the exact eight fermion vertex also has another physically interesting expansion. In particular, $\mathcal{F}_4(\vec{X})$ is a periodic function of ω and σ with period 2π in each variable. Hence it will have a double Fourier expansion of the form,

$$\mathcal{F}^{(4)}(\vec{X}) = \sum_{k=-\infty}^{+\infty} \sum_{q=-\infty}^{+\infty} \tilde{\mathcal{F}}_{q,k}(|\phi|) \exp(-iq\omega + ik\sigma)\,. \tag{22}$$

We will refer to this as the IIB expansion for reasons which will become clear momentarily. It is straightforward to extract the leading behaviour of the coefficients $\mathcal{F}_{q,k}$ in the regime where the compactification radius is large and we are "near four dimensions". For $\beta|\phi| \gg 1$ we find,

$$\tilde{\mathcal{F}}_{q,k} \sim \beta^{-5} g^{12} |\phi|^{-12} \left(\beta M(q,k)\right)^{11/2} \exp(-\beta M(q,k))\,, \tag{23}$$

where $M(q,k) = |\phi||k\tau + q|$ is the mass of a BPS state of the four-dimensional theory with electric and magnetic charges q and k, respectively. However, note that the sum in (22) is not restricted to the coprime values of q and k where we expect the existence of a BPS bound state in four dimensions. The exact formulae predict a series of corrections to (23) in inverse powers of $\beta|\phi|$. However, unlike the perturbative corrections to the IIA series (20), these corrections form an infinite series.

These effects have simple interpretation in string theory. We start by returning to $\mathcal{N} = 4$ SUSY Yang-Mills in four-dimensional Minkowski space which lives on the world volume of two D3-branes of the IIB theory on $\mathbb{R}^{9,1}$. As above the $\mathcal{N} = 4$ theory has BPS states with coprime electric and magnetic charges q and k

which are realized as (q,k)-strings stretching between the D3-branes. We will now go to euclidean space and compactify the theory on a circle of circumference β with supersymmetry preserving boundary conditions. For $\beta|\phi| \gg 1$, the compactification radius is large and, in contrast to our earlier discussion, it is not appropriate to go to the IIA picture via T-duality. Rather in the IIB picture we now have (q,k)-strings with euclidean worldsheets of finite area $\beta M(q,k)$ which stretch between the two D3s and also wrap the compact dimension. These configurations have finite action and contribute as instantons with the same exponential supression proportional to $\exp(-\beta M(q,k))$ as the terms in (23). However, so far we have only discussed coprime values of k and q. In the more general case suppose k and q have maximum common divisor $l > 1$, then we have $M(q,k) = lM(q/l, k/l)$ and we can obtain a configuration with action $\beta M(q,k)$ by wrapping the worldsheet of a $(q/l, k/l)$ string stretched between the two D3-branes l times around the compact dimension. The usual rules of D-brane instanton calculus [23] indicate that such configurations should still be invariant under half the supersymmetry algebra and thus contribute to our eight fermion vertex. Note that similar configurations involving more than one wrapped string have additional zero modes which would prevent them from contributing.

We will now interpret this IIB picture in the context of the gauge theory path integral. As usual the euclidean path integral with periodic boundary conditions can be interpreted as a trace over the Hilbert space of the four-dimensional theory with an insertion of $(-1)^F$, where F is the fermion number operator. We will apply this interpretation to the path integral formula for the correlation function $\mathcal{G}^{(8)}$ appearing in (10) which corresponds an eight fermion vertex of the form (9). Taking account of surface terms we find that,

$$\begin{aligned}\mathcal{G}^{(8)}(x_1,\ldots,x_8) &= \left\langle \prod_{A=1}^{4} \rho_1^A(x_{2A})\rho_2^A(x_{2A-1}) \right\rangle \\ &= \mathrm{Tr}\left[(-1)^F \prod_{A=1}^{4} \rho_1^A(x_{2A})\rho_2^A(x_{2A-1}) \exp\left(-\beta H - i\omega Q + i\sigma K\right)\right],\end{aligned} \tag{24}$$

where H is the four-dimensional hamiltonian and Q and K are the electric and magnetic charge operators, respectively (which have integer eigenvalues q and k). This quantity can be thought of as a generalized index. Unlike a conventional Witten index, the fermionic insertions ensure that states of non-zero energy contribute to the trace. In particular, the BPS states discussed above then contribute to the trace (24) with the exponential supression $\exp(-\beta M(q,k) - iq\omega + ik\sigma)$. We will evaluate these contributions explicitly in [7] and find that the eight fermionic insertions in (24) effectively act as a projection operator onto the BPS sector of the Hilbert space so that *only* BPS states contribute to the trace at leading semiclassical order.

The appearance of the electric surface term $-iq\omega$ in the exponent of (24) has an interesting explanation which we will now sketch. In the four-dimensional theory, one is free to make the gauge choice $A_0 = 0$ and in fact this is the convenient choice for discussing the semiclassical quantization of monopoles and dyons. After compactification this leads to a puzzle because it appears that we have lost the fluctuating degree of freedom corresponding to the Wilson line in our previous approach! This is not quite correct however because, to make this gauge choice consistently in the hamiltonian we still have to impose Gauss' law as a constraint. This can be accomplished by introducing a Lagrange multiplier field ω in close analogy to the way σ first appears as a Lagrange multiplier for the Bianchi identity. In the latter case the result is the magnetic surface term $ik\sigma$ considered above, in the former case we obtain $-iq\omega$ where q is the electric charge (1). Note that ω is a free periodic scalar field and the period is 2π because of electric charge quantization. Hence we identify ω with the Wilson line we have denoted by the same letter above. One indication that this identification is correct is that it renders the exponent in (22) invariant under electromagnetic duality. In particular the combination $iq\omega - ik\sigma$ can be written in terms of the $\mathrm{SL}(2,\mathbb{Z})$ vectors $\vec{\sigma}$ and $\vec{q}$ introduced above as $\epsilon_{RS}\sigma_R q_S$ where $R, S = 1, 2$ and ϵ_{RS} is the invariant symplectic form of $\mathrm{SL}(2,\mathbb{Z})$.

So far we have considered instanton expansions arising in two different limits of the theory. In each case there are perturbative corrections to the leading instanton contributions and the instanton series only makes sense when these are small. In this sense, the IIA expansion (20) is valid for $g^2 \ll 1$ while the IIB expansion (23) holds for $\beta|\phi| \gg 1$. Interestingly, the regimes in which these two series are valid have some overlap and, in the following, we will investigate the relation between them in their regime of common validity. In fact when we impose both $g^2 \ll 1$ *and* $\beta|\phi| \gg 1$, we will also be able to compare both expansions directly with first principles semiclassical calculations [7]. As a preliminary, we need to extract a prediction for the coefficient function $\mathcal{V}$ of the instanton induced vertex (9). This requires us to find the relation between the Majorana fermions appearing in our exact eight fermion vertex and the four-dimensional Weyl fermions ρ^A_δ and $\bar{\rho}^A_{\dot{\delta}}$ which appear in (9). Following the discussion above, the components of the right handed Weyl fermions $\bar{\rho}^A_{\dot{\delta}}$ are identified with complex combinations of Majorana fermions which transform with $U(1)_N$ charge $-1/2$. The overall normalizations of the two sets of fermions are related as $\psi \sim \bar{\rho}\,\sqrt{\beta/g^2}$. For $\beta|\phi| \gg 1$ and $g^2 \ll 1$ we have $A \gg X_7, X_8$ and we find,

$$\mathcal{V}(\sigma,\omega,|\phi|) = \left(\frac{\beta}{g^2}\right)^4 \sum_{n_1,n_2=-\infty}^{+\infty} \frac{A^4}{\left[(X_8 - 2n_1\Omega_1 - 2\kappa n_2\Omega_2)^2 + (X_7 - 2n_2\Omega_2)^2 + A^2\right]^7}. \tag{25}$$

Perturbative corrections to the IIA instanton series (20) can be neglected if $g^2 \ll 1$. If we also impose $\beta|\phi| \gg 1$, the instanton action $S_{k,n}$ appearing in the (20) can be approximated as,

$$S_{k,n} = k\beta M(0,1) + \frac{1}{2}k\Lambda\,\frac{(\omega - 2\pi n)^2}{\beta} + \mathrm{O}\left(\beta^{-3}|\phi|^{-3}\right), \tag{26}$$

where $\Lambda = M(0,1)/|\phi|^2$. The resulting IIA series for $\mathcal{V}$ is,

$$\mathcal{V}_{\mathrm{IIA}} = \sum_{k=1}^{\infty}\sum_{n=-\infty}^{+\infty} \mathcal{V}_{k,n}\exp ik\left(\sigma + n\theta + \frac{\theta\omega}{2\pi}\right) \tag{27}$$

with

$$\mathcal{V}_{k,n} = \left(\frac{\beta}{g^2}\right)^9 \frac{k^6}{(\beta M(0,1))^3}\,\exp\left(-\beta k M(0,1) - \frac{1}{2}k\Lambda\frac{(\omega-2\pi n)^2}{\beta}\right). \tag{28}$$

As above, only the $n = 0$ term survives in the $D = 3$ limit. This term reproduces the k instanton effect calculated in the three-dimensional theory in [3].

On the other hand, if $\beta|\phi| \gg 1$, perturbative corrections (in inverse powers of $\beta|\phi|$) to the IIB series (23) can be neglected. If we impose $g^2 \ll 1$, we may also approximate the BPS mass formula as,

$$\beta M(q,k) = k\beta M(0,1) + \frac{\beta\left(q + \theta/2\pi\right)^2}{2k\Lambda} + \mathrm{O}\left(g^6\right). \tag{29}$$

The resulting IIB series for $\mathcal{V}$ is,

$$\mathcal{V}_{\mathrm{IIB}} = \sum_{k=1}^{\infty}\sum_{q=-\infty}^{+\infty} \tilde{\mathcal{V}}_{q,k}\exp(ik\sigma - iq\omega) \tag{30}$$

with

$$\tilde{\mathcal{V}}_{q,k} = \left(\frac{\beta}{g^2}\right)^8 \beta\,\frac{k^{11/2}}{(\beta M(0,1))^{5/2}}\,\exp\left(-\beta k M(0,1) - \frac{\beta\left(q+\theta/2\pi\right)^2}{2k\Lambda}\right). \tag{31}$$

Thus we have two alternative series for $\mathcal{V}$, both of which are both valid (in the sense that power law corrections are small) when $\beta|\phi| \gg 1$ and also $g^2 \ll 1$. However, we must also consider the convergence of the series themselves. In fact the sum over winding number n in the IIA series (27) converges rapidly only when $\beta|\phi|g^2 \ll 1$. On the other hand, rapid convergence of the sum over electric charge q in the IIB series (30) requires $\beta|\phi|g^2 \gg 1$. Hence the two approximate series (27) and (30) are useful in complimentary regions of parameter space.

It is interesting to investigate the physical content of these weak coupling predictions. In particular we can compare the detailed expressions (28) and (31), to a semiclassical instanton calculation on $\mathbb{R}^3 \times S^1$, where the collective coordinates of the monopole are treated in the moduli space approximation [24]. Details of these calculations will be given in [7]. However it is easy to understand the exponents appearing in the approximate IIA and IIB series. For this purpose it is be sufficient to recall that, in addition to its position in three-dimensional space, a single BPS monopole in the four-dimensional theory has an additional modulus describing its orientation in the global part of the unbroken U(1) gauge group. This collective coordinate is a periodic variable $\chi \in [0, 2\pi]$ and its dynamics are described, in the moduli space approximation, by the effective lagrangian $L_\chi = \Lambda\dot{\chi}^2/2$. Here the dot denotes differentiation with respect to time and, as in (29) above, we have $\Lambda = M(0,1)/|\phi|^2$. The quantity Λ is the monopole's moment of inertia with respect to global gauge rotations. The corresponding hamiltonian is $H_\chi = Q^2/2\Lambda$, where Q, the canonical momentum conjugate to χ, is identified with the electric charge. In the semiclassical limit of the $\mathcal{N} = 4$ theory, we may simply proceed by quantizing this system. The condition that the resulting wavefunctions are single valued on S^1 naturally leads to the integer quantization of electric charge. After continuation to compact euclidean time, the resulting energy eigenstates contribute terms of order $\exp(-\beta q^2/2\Lambda)$ to the partition function $\mathrm{Tr}(\exp(-\beta H_\chi))$. We will call these states "momentum modes" as they are characterised by a definite integer value of the conjugate momentum q. In spacetime these states correspond to an infinite tower of BPS dyons which carry magnetic charge one and electric charge $q \in \mathbb{Z}$. As $M(q,1) = M(0,1) + q^2/2\Lambda + O(g^6)$ these states, saturate the Bogomonl'nyi bound up to higher order corrections in g^2. Including the shift in the electric charge due to non-zero θ [25], we find contributions with the characteristic exponential supression of the $k = 1$ terms in the approximate IIB expansion (31). In [7] we will interpret the corresponding coefficients of these terms, for all electric and magnetic charges, as the bulk contribution to the L^2-index theory required to count BPS (q,k)-dyons in the four-dimensional $\mathcal{N} = 4$ theory.

In fact we can also find a corresponding weak coupling interpretation for the $k = 1$ terms in the approximate IIA series (28). As above the semiclassical contributions are proportional to the quantum mechanical partition function $\mathrm{Tr}(\exp(-\beta H_\chi))$. Rather than evaluating this trace directly in canonical quantization, we may instead consider the corresponding quantum mechanical path integral in compact euclidean time $\tau \in [0, \beta]$,

$$\mathrm{Tr}(\exp(-\beta H_\chi)) = \int [d\chi(\tau)] \exp\left(-\int_0^\beta d\tau\, L_\chi\right). \tag{32}$$

As the dimensionless parameter $\Lambda\beta$ is very large at weak coupling we may evaluate this path integral in the semiclassical approximation. This requires us to sum over classical paths $\chi(\tau)$ with periodic boundary conditions $\chi(\tau + \beta) = \chi(\tau) + 2\pi n$ for integer n. Hence the admissable classical paths are $\chi(\tau) = 2\pi n\tau/\beta$. As these paths describe the worldline of a point particle of mass Λ winding n times around the target circle, we will call these configurations "winding modes". The classical action of a winding mode is precisely $2\Lambda\pi^2 n^2/\beta$. Adding in the action of the static monopole

$\beta M(0,1)$ we reproduce the exponential supression of the $k=1$ terms in (28) for the case $\omega=0$. In spacetime, these winding modes correspond to dyon-like classical solutions which spin in the internal U(1). In particular, they complete exactly n orbits of U(1) in the periodic euclidean time interval $[0,\beta]$. Finally note that because the collective coordinate χ parametrises global U(1) gauge transformations, the τ-dependence is identical to the x_0 dependence of a large gauge transformation of winding number n acting on the monopole. This agrees perfectly with the identification, given above, of the summation variable n appearing in the IIA series as the winding number of a large gauge transformation.

In fact it is well known that the semiclassical approximation to the path integral (32) is actually exact [26]. The resulting sum over winding modes coming from the r.h.s. of (32) is equal to the sum over momentum modes appearing in the trace on the l.h.s. As a consequence, the the approximate IIA/IIB series (27) and (30) are actually exactly equal: $\mathcal{V}_{\rm IIA}=\mathcal{V}_{\rm IIB}$. In particular we have,

$$\sum_{n=-\infty}^{+\infty}\mathcal{V}_{k,n}\exp\left(ik\theta\frac{(\omega+2\pi n)}{2\pi}\right)=\sum_{q=-\infty}^{+\infty}\tilde{\mathcal{V}}_{q,k}\exp(-iq\omega)\,. \tag{33}$$

This equality follows immediately from the Poisson resummation formula,

$$\sum_{q=-\infty}^{+\infty}\exp\left[-i\pi Bq^2+2\pi isq\right]=\frac{1}{\sqrt{iB}}\sum_{n=-\infty}^{n=+\infty}\exp\left[\frac{i\pi}{B}(s+n)^2\right] \tag{34}$$

with $B=\beta/2\pi i\Lambda$ and $s=\omega/2\pi$. Of course this Poisson-Lie duality between momentum and winding modes is nothing other than the T-duality between the IIB and IIA pictures described above. As expected, the IIB series is convergent near four dimensions ($\beta|\phi|g^2\gg 1$) and the IIA series is convergent near three dimensions ($\beta|\phi|g^2\ll 1$).

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